An Analytic Series Solution for H-Plane Waveguide T-Junction

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***Abstract*—A solution for the H-plane waveguide T-junction is obtained in analytic series form. A Fourier-Transform technique is employed to express the scattered field in the spectral domain in terms of parallel-plate waveguide modes. The boundary conditions are enforced to obtain simultaneous equations for the transmitted field. The simultaneous equations are solved to obtain the transmission and reflection coefficients in simple series forms. Comparisons between our solution and other existing results show excellent agreements.**

I. Introduction

The PROBLEM of the rectangular waveguide T-junction has been still considered by many investigators [1]–[3] ever since Marcuvitz [4] first obtained the approximate solution based on the variational principle. The motivation of this letter is to formulate the problem of scattering in the T-junction waveguide using the Fourier transform, thus obtaining a series solution in an analytic form. Using the Fourier transform and the residue calculus, we obtain (2.8), (2.9), and (2.10), which are the simple series representations for the transmission and reflection coefficients in the next section. A brief summary of the scattering analysis is given in the Conclusion.

II. Reflection and Transmission Coefficients

The waveguide T-junction of the H-plane is shown in Fig. 1. An electromagnetic wave $E_y^{i\mathrm{I}}(x,z)$, which is transverse-electric (TE) to the x-axis, is incident on the junction. Here $\exp(-jwt)$ time factor is suppressed. Then, in region (I)$(-b<z<0)$, the total electric field consists of the incident and scattered fields $E_y^{\mathrm{I}}(x,z)$, which are, respectively, written as

$$E_y^{i\mathrm{I}}(x,z) = e^{jk_{xs}x}\sin k_{zs}(z+b)$$
$$E_y^{\mathrm{I}}(x,z) = 1/(2\pi)\int_{-\infty}^{\infty}\tilde{E}_y^{\mathrm{I}}(\zeta)\sin(\kappa_1 z)e^{-j\zeta x}d\zeta,$$

where

$$\kappa_1 = \sqrt{k_1^2-\zeta^2}$$
$$k_{zs} = \frac{s\pi}{b} \qquad s:\ \text{integer}$$
$$k_{xs} = \sqrt{k_1^2-k_{zs}^2}.$$

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Since $H_x(x,z) = -1(j\omega\mu)\partial E_y/\partial z$, the corresponding x components of the incident and the scattered H-fields are

$$H_x^{i\mathrm{I}}(x,z) = \frac{-k_{zs}}{j\omega\mu}e^{jk_{xs}x}\cos k_{zs}(z+b)$$
$$H_x^{\mathrm{I}}(x,z) = \frac{-1}{j\omega\mu}\int_{-\infty}^{\infty}\frac{\kappa_1}{2\pi}\tilde{E}_y^{\mathrm{I}}(\zeta)\cos(\kappa_1 z)e^{-j\zeta x}d\zeta.$$

In region (II), $(-a<x<a, z<-b)$ the total transmitted field may be represented with a summation of parallel-plate waveguide modes,

$$E_y^{\mathrm{II}}(x,z) = \sum_{m=1}^{\infty} c_m \sin a_m(x+a)e^{-j\xi_m z}, \qquad (2.1)$$

where

$$a_m = m\pi/(2a)$$
$$\xi_m = \sqrt{k_1^2-a_m^2}.$$

The corresponding magnetic field is

$$H_x^{\mathrm{II}}(x,z) = \sum_{m=1}^{\infty} c_m\frac{\xi_m}{\omega\mu}\sin a_m(x+a)e^{-j\xi_m z}.$$

To determine unknown coefficient c_m, it is necessary to match the boundary conditions of tangential E- and H-field continuities. The tangential E-field continuity at $z=-b$ yields

$$E_y^{\mathrm{I}}(x,-b) = E_y^{\mathrm{II}}(x,-b), \qquad |x|<a,$$
$$= 0, \qquad |x|>a.$$

Taking the Fourier transform on both sides of the previous equation, we get

$$\int_{-\infty}^{\infty}E_y^{\mathrm{I}}(x,-b)e^{j\zeta x}dx = \int_{-a}^{a}E_y^{\mathrm{II}}(x,-b)e^{j\zeta x}dx. \qquad (2.2)$$

Substituting (2.1) into (2.2), and performing integration with respect to x, we obtain

$$\tilde{E}_y^{\mathrm{I}}(\zeta)\sin(-\kappa_1 b) = \sum_{m=1}^{\infty} c_m\frac{a_m e^{j\xi_m b}}{(\zeta^2-a_m^2)}\left[e^{j\zeta a}(-1)^m - e^{-j\zeta a}\right]. \qquad (2.3)$$

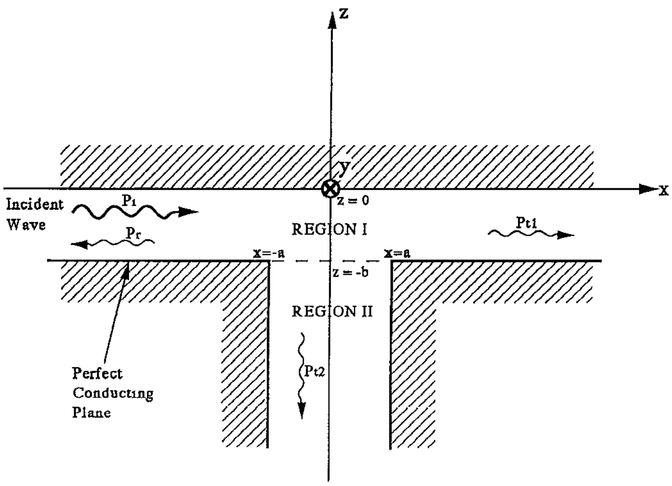

Fig. 1. Scattering geometry of H-plane waveguide T-junction.

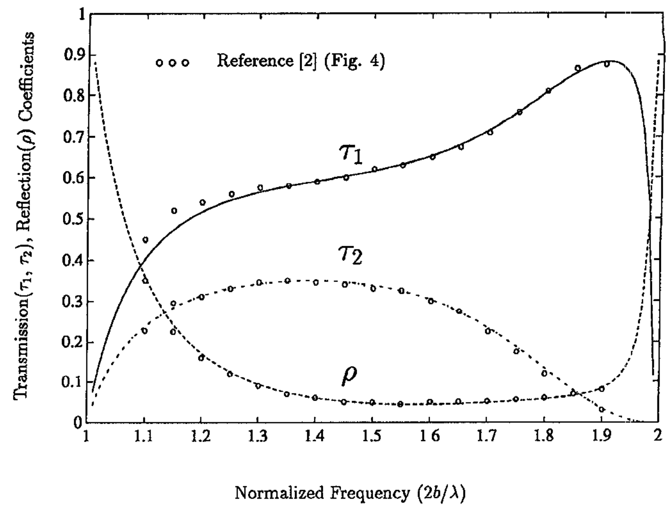

Fig. 2. Behavior of transmission and reflection coefficients versus $2b/\lambda$ when $b = 2a(k_1 = 2\pi/\lambda)$.

Second, the tangential H-field continuity along $(-a < x < a, z = -b)$ gives

$$H_x^{iI}(x-b,) + H_x^{I}(x,-b) = H_x^{II}(x,-b)$$

$$k_{zs}e^{jk_{xs}x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} \kappa_1 \cos(\kappa_1 b)\tilde{E}_y^{I}(\zeta)e^{-j\zeta x}d\zeta = \sum_{m=1}^{\infty} c_m(-j\xi_m)e^{j\xi_m b}\sin a_m(x+a). \qquad (2.4)$$

Substituting (2.3) into (2.4), we obtain

$$k_{zs}e^{jk_{xs}x} - \sum_{m=1}^{\infty} c_m \frac{a_m e^{j\xi_m b}}{2\pi}\int_{-\infty}^{\infty}\kappa_1 \cdot \cot(\kappa_1 b)\left[\frac{(-1)^m e^{j\zeta a} - e^{-j\zeta a}}{\zeta^2 - a_m^2}\right]e^{-j\zeta x}d\zeta = -j\sum_{m=1}^{\infty} c_m\xi_m e^{j\xi_m b}\sin a_m(x+a).$$

In order to determine the coefficient c_m, we multiply the previous equation by $\sin a_n(x+a)$ and integrate the both sides with respect to x from $-a$ to a, then we obtain

$$\frac{k_{zs}a_n}{k_{xs}^2 - a_n^2}\left[(-1)^n e^{jk_{xs}a} - e^{-jk_{xs}a}\right] = \sum_{m=1}^{\infty} c_m \frac{a_m a_n e^{j\xi_m b}}{2\pi} I_{nm} - jc_n\xi_n a e^{j\xi_n b}, \qquad (2.5)$$

where

$$I_{nm} = \int_{-\infty}^{\infty} \kappa_1 \cot(\kappa_1 b) \cdot \frac{\left[(-1)^m e^{j\zeta a} - e^{-j\zeta a}\right]\left[(-1)^n e^{-j\zeta a} - e^{j\zeta a}\right]}{(\zeta^2 - a_m^2)(\zeta^2 - a_n^2)}d\zeta.$$

Utilizing the technique of the contour integration, we obtain

$$I_{nm} = h_m\delta_{nm} + r_{nm}, \qquad (2.6)$$

where

δ_{nm} : Kronecker delta

$$h_m = \frac{2\pi a\xi_m}{a_m^2 \tan(\xi_m b)}$$

$$r_{nm} = \sum_{l=1}^{\infty} \cdot \frac{-j4\pi(l\pi/b)^2\left[1 - (-1)^m e^{j2\sqrt{k_1^2 - (l\pi/b)^2}a}\right]}{\sqrt{k_1^2 - (l\pi/b)^2}b\left[\xi_m^2 - (l\pi/b)^2\right]\left[\xi_n^2 - (l\pi/b)^2\right]}.$$

Substituting (2.6) into (2.5), we obtain the simultaneous equations for c_m, which may be given as

$$C = (U - P)^{-1}Q = Q + PQ + P^2Q + \cdots, \qquad (2.7)$$

where C is the column vector of elements c_m, U the identity matrix, P the full matrix of elements p_{nm}, and Q the column vector of elements of q_n. The expressions of p_{nm} and q_n are given as

$$p_{nm} = \frac{-a_n a_m \sin(\xi_n b)e^{j\xi_m b}r_{nm}}{2\pi a\xi_n}$$

$$q_n = \frac{k_{zs}a_n \sin(\xi_n b)\left[(-1)^n e^{jk_{xs}a} - e^{-jk_{xs}a}\right]}{\xi_n a(k_{xs}^2 - a_n^2)}.$$

Once c_m is determined, it is possible to evaluate the transmitted and reflected fields as follows.

In region (I), the total scattered field is

$$E_y^{I}(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{E}_y^{I}(\zeta)\sin(\kappa_1 z)e^{-j\zeta x}d\zeta,$$

where

$$\tilde{E}_y^{I}(\zeta) = \sum_{m=1}^{\infty} c_m \frac{-a_m e^{j\xi_m b}}{\sin(\kappa_1 b)(\zeta^2 - a_m^2)}\left[e^{j\zeta a}(-1)^m - e^{-j\zeta a}\right].$$

By use of residue calculus, we evaluate total transmitted and reflected fields at $x = \pm\infty$ in region (I) such as

$$E_y^{\mathrm{I}}(\pm\infty, z) = \sum_v K_v^{\pm} \sin k_{zv}(z+b) e^{\pm j k_{xv} x},$$

where

$$1 \leq v < \frac{k_1 b}{\pi} \qquad v: \text{ integer } (1, 2, 3, \cdots)$$

$$k_{zv} = \frac{v\pi}{b}$$

$$k_{xv} = \sqrt{k_1^2 - k_{zv}^2}$$

$$K_v^{\pm} = \sum_{m=1}^{\infty} c_m \frac{j a_m k_{zv} e^{j\xi_m b}}{k_{xv} b (k_{xv}^2 - a_m^2)} \left[e^{\mp j k_{xv} a}(-1)^m - e^{\pm j k_{xv} a}\right].$$

Let the time-averaged incident, transmitted, and reflected powers be denoted by P_i, P_{t1}, P_{t2}, and P_r as shown in Fig. 1, then

$$\tau_1 = P_{t1}/P_i = \left|1 + K_s^+\right|^2 + \frac{1}{k_{xs}} \sum_{v \neq s} k_{xv} \left|K_v^+\right|^2 \quad (2.8)$$

$$\rho = P_r/P_i = \frac{1}{k_{xs}} \sum_v k_{xv} \left|K_v^-\right|^2 \quad (2.9)$$

$$\tau_2 = P_{t2}/P_i = \frac{2a}{k_{xs} b} \sum_m \xi_m |c_m|^2, \quad (2.10)$$

where

$$1 \leq m < \tfrac{2ak_1}{\pi}, \quad m: \text{ integer},$$
$$1 \leq v < \tfrac{k_1 b}{\pi}, \quad v: \text{ integer}.$$

We evaluate τ_1, τ_2, and ρ for $b = 2a$ and $1 < (k_1 b/\pi) < 2$ in Fig. 2 and compare them with Fig. 4 in [2] (or Fig. 2 in [3]), thus confirming excellent agreements. We also evaluate the phases of the scattering parameters as defined by (12a) and (12b) in [2]. Our results are identical with Fig. 7 in [2] so that we do not show the comparison here. Note that our results used in the previous comparison satisfy the power conservation check $|1.0 - (\tau_1 + \tau_2 + \rho)| < 10^{-6}$ when $m = 6$. Our computational experience shows that the number of modes m to be used for the evaluation of c_m in (2.7) must be at least $2ak_1/\pi$ to achieve the numerical accuracy. This means that the series expression c_m given in (2.7) is very fast convergent and numerically efficient.

III. Conclusion

The problem of waveguide T-junction of H-plane is analytically solved. The Fourier transform method is used to obtain the solution in a simple analytic series form. The comparisons between ours and other existing results show excellent agreements.

References

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